PLACING N NON DOMINATING QUEENS ON THE N x N CHESSBOARD

PART I : SOLUTIONS FOR N $\leq$ 18

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09/06/2011

This article was published in May 2011 on the site of the "Fédération Française des Jeux Mathématiques" (French Federation of Mathematical Games): FFJM.

• Statement of the problem

The problem consists in placing N queens on the N x N board so as to maximize U(N), the number of unattacked (free) squares.

• History of the problem

As far as we know this problem of placement was first mentionned by Walter W. ROUSE BALL in his book "Mathematical Recreations and Problems of Past and Present Times", Mac Millin, 3rd edition, 1896 (this book was revised and enlarged by H. S. M.COXETER and republished by Dover ed ; the 13th edition appeared in 1987). W. ROUSE BALL wrote : "Captain TURTON first called my attention … to place 8 queens on a chessboard, so as to command the fewest possible squares. For example : if eight queens are placed on cells 21, 22, 62, 71, 73, 77, 82, 87, eleven cells of the board will not be in check. The same number can be obtained by other arrangements. Is it possible to place the eight queens so as to leave more than eleven cells out of check? I have never succeeded in doing so, neither in showing that it is impossible to do it". Here is the placement of ROUSE BALL :

1	2	3	4	5	6	7	8	
			U	U	U		U	1
Q	Q							2
			U		U		U	3
				U			U	4
					U		U	5
	Q							6
Q		Q				Q		7
	Q					Q		8

We will see further that U(8) = 11 is actually the maximal number of unattacked squares, and there are 7 « basic » solutions (not counting placements obtained by symmetry or rotation).

Other authors early mentionned this problem :

- Wilhelm E. AHRENS, "Mathematische Unterhaltungen And Spiele", Teubner, Leipzig, 1901 (it can now be read online).
- Italo GHERSI, « Mathematica dilettevole curiosa », Hoepli, Milan 1913 (reedited by Hoepli in 1990).
- Of course, Henry E.DUDENEY dealt with this problem : see « Amusements in mathematics », Nelson and sons, 1917 (reedited many times by Dover from 1958), problem n°316 "The Amazons". DUDENEY mentions :"I will hazard the statement that 8 queens cannot be places on the chessboard so as to leave more than 11 squares unattacked. It is true that we have no rigid proof of this yet, but I have entirely convinced myself of the truth of the statement. There are at least 5 different ways [to do so]".

Actually we illustrate further the 7 different basic ways to do so.

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“Basic” means that all the other solutions can be deduced from these 7 basic ones, by symmetry and/or rotation.

During the 1970’s Stephen AINLEY found placements giving the results below :

N	4	5	6	7	10	12	14	18	19	22
U (N)	1	3	5	7	22	36	56	111	132	186

These values were later shown to be optimal for N≤ 16, and are presumed optimal for N ≥17. S. AINLEY published the book : « Mathematical Puzzles », G. BELL and sons, U.K. 1977 and contributed several times to Martin GARDNER’s column in the « Scientific American » (June 1972, Feb 1978, etc).

In the Argentinian review « El Acertijo », # 6 (1993) two solutions were published :
For N = 9, U(9) = 18, by Diego BRACAMONTE.
For N = 11, U(11) = 30, by Rodolfo KURCHAN.
Further publications in « El acertijo » appeared in : # 13 (1994) and # 15 (1996).
Also in : « Los Acertijeros Boletin » R.KURCHAN, # 26 (1997).

From 1995, Mario VELUCCHI of Pisa (Italy) systematically studied the problem ; He established that the known values of U(N) for N<17, are optimal ; He also gave placements believed to be optimal for 17 ≤ N ≤ 22. He later continued this work for 23 ≤ N ≤ 30, together with Johan CLAES and Mioshi NAGAI (Japan), but whithout proof of optimality. He also gave the numbers of equivalent optimal solutions (« variants ») for N ≤ 16, cf the french review “Jouer Jeux Mathématiques”, n°19.

These results were to be published in the J.R.M, # 28/4 (1996-97) … but did not appear ; moreover most of them are not (or no longer) accessible on the Internet (in particular the corresponding placements of the N queens leaving U(N) unattacked squares) :
Only partial result can be found in N.J.A SLOANE (on-line Encyclopedia of Integer Sequences ; OEIS Foundation : A001366 is a table listing UCN) for N = 1 to 30, (see also « Revision history for A001366 ») but these pages do not give the corresponding placements of the queens.

• Aim of this article

Our intention in this article is to fill the gaps and investigate placements for larger boards :

- Show the optimal placements of N Queens on the N x N board, with all their possible variants for N ≤ 16 (we recreated results of M.VELUCCHI et al.).

- Show some best known placements for 17≤ N ≤ 30 with several variants.
  Actually boards with size N ≥ 19 are treated in part II.

- Extend the problem for 31 ≤ N ≤ 45 : in part II we give « good » placements (with variants) ; Most are thought to be optimal : a challenge for future followers !

- Provide an « asymptotic » study for the behaviour of several heuristics.

N.B : this research work was done in a spirit of recreational mathematics (only using computer-aided human intelligence !). This optimization problem can be modelled and solved using a 0-1 linear program (cf Part II) for N not too large.

N = 4 : THE 25 SOLUTIONS : U(4) = 1

Q		Q	
Q		Q	
	U		

1

	Q		
		Q	
U			
		Q	Q

2

		U	
Q			
			Q
	Q		Q

3

Q			
Q	Q		
			U
Q			

4

			U
Q			
Q		Q	
	Q		

5

	Q	Q	
Q	Q		
			U

6

Q		Q	
		Q	
			Q
	U		

7

SYMETRICAL

	Q	Q	
Q			
Q			
			U

8

U			
	Q		Q
	Q	Q	

9

Q			Q
			Q
Q			
		U	

10

			Q
U			
			Q
	Q		Q

11

Q		Q	
Q	Q		
			U

12

	Q		
			Q
		Q	Q
U			

13

Q	Q		
			U
	Q		
Q			

14

	Q		
			U
Q	Q		
Q			

15

		U	
Q			
	Q		Q
	Q		

16

SYMETRICAL

		Q	
		Q	
Q	Q		
			U

17

Q			
	Q		Q
Q			
		U	

18

	Q	Q	
	Q		
		Q	
U			

19

			Q
			Q
U			
		Q	Q

20

	Q		
		Q	
Q	Q		
			U

21

	Q		Q
		Q	
U			
			Q

22

Q		Q	
Q			
			U
	Q		

23

SYMETRICAL

Q			
		U	
Q			
	Q		Q

24

SYMETRICAL

U			
		Q	
	Q		Q
		Q	

25

- There are 4 symetrical solutions : n°8, 17, 24, 25.

- There is only one solution for which the unattacked (« free ») square is not on an edge of the board : n°24. [as pointed out by Mr Alain Joisel).

N = 5 : THE SOLUTION (UNIQUE) ; U(5) = 3

		Q	Q	
Q			Q	
Q				
				U
	U			U

N = 6 : THE 3 SOLUTIONS ; U(6) = 5

Q				Q	
Q	Q			Q	
	Q				
					U
	U				U
	U	U			

	Q		Q	
Q		Q		
				U
		U		U
Q	Q			
		U		U

Q		Q	Q		
			Q		
Q		Q			
				U	
	U				U
	U			U	

N = 7 : 38 SOLUTIONS ; U(7) = 7

This is a very particular case since U(N) = N for N = 7 ; So one can deduce from any solution S another one : S*. Just replace each queen of S by an unattacked (« free ») square in S* ; reciprocally, each free square of S is replaced by a queen in S*. Hence we show below only 19 of the 38 solutions : the 19 solutions that are not shown here can be deduced from the 19 solutions shown below, using the transformation S → S*.

		Q	Q		Q	
Q			Q			
						U
Q		Q				
				U		
	U					U
	U			U		U

1

Q						Q
	Q					
			U		U	
		U		U		
Q						Q
		U	U	U		
Q						Q

2

Q		Q				Q
				U		
	U		U		U	
				U		
Q						Q
			U	U		
Q						Q

3

Q			Q	Q		Q
			Q			
Q				Q		
		U				
	U				U	
		U				
	U	U			U	

4

Q		Q	Q			
			Q			
Q		Q				Q
				U		
	U				U	
	U			U		
	U				U	

5

Q			Q	Q		
			Q			
Q				Q		
			Q			
	U				U	
		U				U
	U	U			U	

6

			Q	Q		
Q				Q		
Q	Q					
	Q					
					U	U
		U			U	U
		U				U

7

				Q		
Q				Q		
Q	Q			Q		
	Q					
					U	
		U			U	U
		U	U			U

8

	Q		Q			
					U	U
Q	Q		Q			
Q	Q					
				U		U
					U	
		U				U

9

Q					Q	
Q	Q				Q	
	Q	Q				
				U		U
						U
			U			U
			U	U		

10

Q					Q	
Q	Q				Q	
	Q					
				U		U
Q						
		U	U			U
			U	U		

11

Q	Q				Q	
Q	Q				Q	
	Q					
						U
						U
		U	U			
		U	U	U		

12

Q	Q			Q	Q	
Q	Q					
	Q					
						U
		U				U
		U	U			
		U	U			

13

Q				Q	Q	
Q	Q				Q	
	Q					
						U
						U
		U	U			U
		U	U			

14

Q					Q	
Q	Q				Q	
	Q					
				U		U
Q						
		U	U			U
			U	U		

15

Q				Q	Q	
Q	Q			Q		
	Q					
		U				U
		U	U			U
		U	U			

16

Q	Q			Q		
Q	Q			Q		
	Q					
					U	
		U				U
		U	U			
		U	U			

17

Q				Q		
	Q			Q		
Q				Q		
	Q					
			U		U	
		U				U
		U	U		U	

18

Q	Q				Q	
	Q	Q			Q	
		Q				
						U
			U			U
			U	U		
			U	U		

19

U	U				U	
	U	U			U	
		U				
						Q
			Q			Q
			Q	Q		
			Q	Q		

19*

An example of the transformation S → S* is given above : 19 → 19*

8 QUEENS ON 8 x 8 BOARD LEAVING U(8) = 11 FREE SQUARES :

DUDENEY'S SOL.

Q				Q	Q		
Q	Q				Q		
Q	Q						
						U	
						U	U
		U				U	U
		U	U				U
		U	U				

Q				Q	Q		
	Q			Q	Q		
Q							
	Q						
			U			U	
		U				U	U
		U	U				U
		U	U			U	

		Q	Q	Q			
						U	U
Q				Q			
Q							
Q		Q					
					U	U	
	U				U	U	U
	U					U	U

SYMETRICAL

ROUSE BALL’S SOL

	Q					Q	
Q		Q				Q	
	Q						
					U		U
				U			U
			U		U		U
Q	Q						
			U	U	U		U

SYMETRICAL

	Q		Q				
					U	U	U
Q			Q				
Q	Q						
Q	Q						
				U	U		U
					U	U	
		U				U	U

				Q	Q		
Q				Q	Q		
Q	Q						
	Q						
						U	
		U				U	U
		U	U			U	U
		U	U				U

Q					Q		
Q	Q				Q		
Q	Q						
	Q						
						U	U
		U				U	U
		U	U				U
		U	U	U			

7 BASIC SOLUTIONS

FOR N = 9 AND FOR N = 10 IS THE SOLUTION UNIQUE :

Q							Q		Q
		U	U	U					
Q							Q		Q
		U		U					
	U		U		U			U	
		U		U				U	
	U		U		U			U	
		U		U					
Q							Q		Q

U(9) = 18 free squares

Q							Q		
Q	Q						Q		
Q	Q	Q							
	Q	Q							
					U	U		U	U
						U		U	U
			U					U	U
			U	U				U	U
			U	U	U				U
			U	U	U	U			

U(10) = 22

Notice the pattern for N = 9 : **N**on **A**djacent **Q**ueens placed in the **4** corners of the board. It is first one of a long series … We call it : **4NAQ**

THE TWO SOLUTIONS FOR N = 11 :

U (11) = 30

Q		Q								Q
				U	U	U	U	U		
Q		Q								Q
				U		U		U		
	U		U		U		U		U	
	U			U		U		U	U	
	U		U		U		U		U	
				U		U		U		
		Q								Q
				U	U	U		U		
Q		Q								Q

Q		Q						Q		Q
				U	U	U				
Q		Q						Q		Q
				U		U				
	U		U		U		U		U	
	U			U		U			U	
	U		U		U		U		U	
				U		U				
Q		Q								Q
				U	U	U				
	U		U		U	U	U		U	

Pattern **4NAQ** appears again (non adjacent queens placed in the four corners). The second solution uses a variant of this pattern.

THE SEVEN SOLUTIONS FOR N = 12 : U(12) = 36

1

Q	Q									Q	Q
	Q									Q	
				U	U	U	U				
		U			U	U			U		
		U	U					U	U		
		U	U	U			U	U	U		
		U	U					U	U		
		U			U	U			U		
				U	U	U	U				
			U	U	U	U	U	U			
Q	Q									Q	Q
Q											Q

2

Q											Q
Q	Q									Q	Q
	Q									Q	
				U	U	U	U				
		U			U	U			U		
		U	U					U	U		
		U	U	U			U	U	U		
		U	U					U	U		
		U			U	U			U		
				U	U	U	U				
			U	U	U	U	U	U			
Q	Q									Q	Q

3

Q	Q										Q
Q	Q									Q	
				U	U	U	U	U			
					U	U	U		U		
		U				U		U	U		
		U	U				U	U	U		
		U	U	U				U	U		
		U	U		U				U		
		U		U	U	U					
			U	U	U	U	U				
	Q									Q	Q
Q										Q	Q

4

Q		Q		Q							
						U		U		U	U
Q		Q		Q							
						U		U	U	U	U
Q		Q		Q							
						U		U		U	U
Q		Q		Q							
						U		U		U	
	U		U		U		U		U		U
						U		U		U	
	U		U		U		U		U		U
	U		U			U		U		U	

This pattern "**1NAQ**" will be also used for N=22

5

Q	Q										
Q	Q	Q									
Q	Q	Q	Q								
	Q	Q	Q								
						U	U	U	U	U	U
							U	U	U	U	U
								U	U	U	U
				U					U	U	U
				U	U					U	U
				U	U	U					U
				U	U	U	U				
				U	U	U	U	U			

6

Q	Q							Q			
Q	Q	Q						Q			
	Q	Q	Q								
		Q	Q								
						U	U		U	U	
							U		U	U	U
				U					U	U	U
				U	U				U	U	U
				U	U	U				U	U
				U	U	U	U				U
				U	U	U	U				
				U	U	U	U		U		

7

Q	Q							Q			
Q	Q	Q						Q			
Q	Q	Q									
	Q	Q									
						U	U		U	U	
							U		U	U	U
									U	U	U
			U	U					U	U	U
			U	U	U					U	U
			U	U	U	U					U
			U	U	U	U	U				
			U	U	U	U	U				

Solutions 1,2 and 3 use the pattern : **4AQ** and are symmetrical.
Solution 5 uses the pattern "**4AQ**" : a group of adjacent queens is placed in a corner.
These patterns will often repeat in the following.

THE SOLUTION FOR N = 13

U(13) = 47

Q		Q								Q		Q
				U	U	U	U	U				
		Q								Q		Q
				U		U		U				
	U		U		U		U		U		U	
	U		U	U		U		U			U	
	U		U		U		U		U		U	
	U			U		U		U			U	
	U		U		U		U		U		U	
				U		U	U	U				
Q		Q										Q
				U	U	U	U	U	U			
Q		Q										Q

pattern : **4NAQ**

THE FOUR SOLUTIONS FOR N = 14 ; U(14) = 56

1 **4NAQ**

Q		Q									Q		Q
				U	U	U	U	U	U				
		Q									Q		Q
				U		U	U		U				
	U		U		U			U		U		U	
	U		U	U					U			U	
	U		U	U						U		U	
	U		U		U					U		U	
	U			U		U			U			U	
	U		U		U		U	U		U		U	
				U		U	U	U	U				
Q		Q									Q		Q
				U	U	U	U	U	U				
Q		Q											Q

second diagonal symmetry

2 **1AQ**

Q	Q	Q											
Q	Q	Q	Q										
	Q	Q	Q	Q									
		Q	Q	Q									
							U	U	U	U	U	U	U
								U	U	U	U	U	U
									U	U	U	U	U
					U					U	U	U	U
					U	U					U	U	U
					U	U	U					U	U
					U	U	U	U					U
					U	U	U	U	U				
					U	U	U	U	U	U			
					U	U	U	U	U	U	U		

U(14) = 56

4 **4AQ**

Q	Q											Q	Q
	Q											Q	
				U	U	U	U	U	U				
		U			U	U	U	U			U		
		U	U			U	U			U	U		
		U	U	U					U	U	U		
		U	U	U	U			U	U	U	U		
		U	U	U					U	U	U		
		U	U			U	U			U	U		
		U			U	U	U	U			U		
				U	U	U	U	U	U				
	Q											Q	
Q	Q											Q	Q
Q													Q

vertical axial symmetry

3 **4AQ**

Q	Q											Q	Q
Q	Q											Q	
				U	U	U	U	U	U				
					U	U	U	U			U		
		U				U	U			U	U		
		U	U						U	U	U		
		U	U	U				U	U	U	U		
		U	U	U					U	U	U		
		U	U			U				U	U		
		U			U	U	U				U		
				U	U	U	U	U					
			U	U	U	U	U	U	U				
Q	Q											Q	Q
Q												Q	Q

First diagonal symmetry

Three (of the four) patterns seen before provide optimal solutions for N = 14 :
4NAQ, 1AQ, 4AQ.

THE THREE SOLUTIONS FOR N = 15

It is remarkable that placing 16 queens [4 tetrads of non adjacent queens] in the four corners of the 15 x 15 board (i.e. 16 queens), leaves 72 squares unattacked .
Hence removing one of the four queens from a tetrad (bottom right for example), also leaves 72 free squares. This leads to four placements of 15 queens on the 15 x 15 board, with U(15) = 72 free « basic » squares. But two of these four placements are symmetrical. Finally one obtains 3 independent solutions with removing the queen in position 1, or 2, or 3 below :

U(15) = 72

Q		Q										Q		Q
				U	U	U	U	U	U	U				
Q		Q										Q		Q
				U		U	U	U		U				
	U		U		U		U		U		U		U	
	U			U		U		U		U			U	
	U		U		U		U		U		U		U	
	U		U	U		U		U		U	U		U	
	U		U		U		U		U		U		U	
	U			U		U		U		U			U	
	U		U		U		U		U		U		U	
				U		U	U	U		U				
Q		Q										1		2
				U	U	U	U	U	U	U				
Q		Q										Q		3

pattern : **4NAQ**

THE SOLUTION FOR N = 16

The optimum placement with U(16) = 82 is unique :

U(16) = 82

				U		U	U	U		U					
Q		Q										Q		Q	
				U	U	U	U	U	U	U					
Q		Q										Q		Q	
				U		U	U	U		U					
	U		U		U		U		U		U		U		U
	U			U		U		U		U			U		
	U		U		U		U		U		U		U		U
	U		U	U		U		U		U	U		U		U
	U		U		U		U		U		U		U		U
	U			U		U		U		U			U		
	U		U		U		U		U		U		U		U
				U		U	U	U		U					
Q		Q										Q		Q	
				U	U	U	U	U	U	U					
Q		Q										Q		Q	

This symmetrical pattern is a variant of **4NAQ**

For N ≥ 17, the following placements of N queens are conjectured to be optimal. So far we have not proved their optimality. We do not know either the numbers of different basic “optimal’ placements.

A SOLUTION FOR N = 17

U(17) = 97

Q		Q												Q		Q
				U	U	U	U	U	U	U	U	U				
Q		Q												Q		Q
				U		U	U	U	U	U		U				
	U		U		U		U	U	U		U		U		U	
	U			U		U		U		U		U			U	
	U		U		U		U		U		U		U		U	
	U		U	U		U		U		U		U	U		U	
	U		U	U	U		U		U		U	U	U		U	
	U		U	U		U		U		U		U	U		U	
	U		U		U		U		U		U		U		U	
				U		U		U		U		U			U	
		Q												●		
				U		U	U	U	U	U		U				
Q		Q												Q		Q
				U		U	U	U	U	U	U	U				
Q		Q												Q		Q

Pattern : **4NAQ**

N.B : if a queen is placed on the black spot above, and placing the 4 groups of queens in the corners of the 18 x 18 board, the solution obtained is not optimal FOR N = 18 : it gives U = 106. Below we give two placements with U(18) = 111, conjectured to be optimal.

TWO SOLUTIONS FOR N = 18

U(18) = 111

Q	Q															Q	Q
Q	Q	Q														Q	
				U	U	U	U	U	U	U	U	U	U				
					U	U	U	U	U	U	U	U			U		
						U	U	U	U	U	U			U	U		
			U				U	U	U	U			U	U	U		
			U	U				U	U			U	U	U	U		
			U	U	U						U	U	U	U	U		
			U	U	U	U				U	U	U	U	U	U		
			U	U	U	U					U	U	U	U	U		
			U	U	U			U				U	U	U	U		
			U	U			U	U	U				U	U	U		
			U			U	U	U	U	U				U	U		
					U	U	U	U	U	U	U				U		
				U	U	U	U	U	U	U	U	U					
	Q	Q														Q	
Q	Q															Q	Q
Q																Q	Q

pattern : **4AQ**

Q	Q	Q	Q														
Q	Q	Q	Q	Q													
	Q	Q	Q	Q													
		Q	Q	Q													
			Q	Q													
									U	U	U	U	U	U	U	U	U
										U	U	U	U	U	U	U	U
					U						U	U	U	U	U	U	U
					U	U						U	U	U	U	U	U
					U	U	U						U	U	U	U	U
					U	U	U	U						U	U	U	U
					U	U	U	U	U						U	U	U
					U	U	U	U	U	U						U	U
					U	U	U	U	U	U	U						U
					U	U	U	U	U	U	U	U					
					U	U	U	U	U	U	U	U	U				
					U	U	U	U	U	U	U	U	U	U			
					U	U	U	U	U	U	U	U	U	U	U		

pattern : **1AQ**

• **ADDENDUM**

Shortly after the publication of this article on the site of the FFJM, Johan Claes sent us three other placements for N = 18. Two of them P1 and P2 are variants of **1AQ** ; 14 queens have the same placements in P1 and P2 : 11, 12, 21, 22, 23, 31, 32, 33, 34, 42, 43, 44, 53, 54. For P1 : place 4 queens on squares 12, 24, 46 and 56. For P2 : place 4 queens on squares 14, 25, 45 and 55. Last : placement P3 uses pattern **4AQ** ; it can be deduced from the first solution shown above for N = 18 by moving the three queens that are placed on the diagonal that is just above the second diagonal of the board (1,7 ; 16,2 ; 17,1) to their symmetrical position with respects to the second diagonal (2,18 ; 3,17 ; 2,18).

• **SUMMARY**

N	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
U (N)	1	3	5	7	11	18	22	30	36	47	56	72	82	97	111
nb sol.	25	1	3	38	7	1	1	2	7	1	4	3	1	1	2+3

N.B : the numbers of "basic" solutions for N $\leq$ 16 were given by Mr.Velucchi, but he did not publish all of them. For N=17 and N = 18 these numbers are only provisional.

END OF PART I

Acknowledgements

We wish to thank Mr Christian BOYER, a French specialist of recreational mathematics, for his constant support and help (esp. for N = 22) during the progression of our work. Also Mr Michel CRITON, president of the "Fédération Française des jeux mathématiques" (French Federation of Mathematical Games) : FFJM, for providing us valuable information.

Bibliography

Relevant references are listed in the paragraph :"History of the problem".